MATH 2040 A Lecture 9 (Oct 6, 2016)

§ Cayley-Hamilton Theorem (textbook § 5.4)

Recall: Given $T : V \to V$, ($\dim V < +\infty$)

a subspace $W \subseteq V$ T-invariant $\iff T(W) \subseteq W$

Examples: $\{0\}$, V, $N(T)$, $R(T)$, E_λ

T-cyclic subspace gen. by $\vec{v} \in V$:

$$W = \operatorname{span}\{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots\ldots\} \subseteq V$$

e.g. $T(\vec{v} + 2T\vec{v}) = T\vec{v} + 2T^2\vec{v} \in W$

Remember: If W is a T-invariant subspace

$\Rightarrow$ do restriction $T|_W : W \to W$ linear on W

Lemma: char. poly of $T|_W$ divides char. poly. of T

Example: Consider

$$T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$$
$$T(f) = f''$$

Q: $x^3 \in P_3(\mathbb{R}) \Rightarrow$ what is the T-cyclic subspace generated by x^3?

$$x^3 \xrightarrow{T} 6x \xrightarrow{T} 0 \xrightarrow{T} 0 \cdots\cdots$$

$W = \mathrm{span}\{x^3, 6x\}$ 2-dim T-invariant $\subseteq P_3(\mathbb{R})$

restriction $\rightsquigarrow$ $T|_W : W \to W$ on 2-dim vector space

Find char. poly. of $T|_W$: $\beta_W = \{x^3, x\}$ basis for W

$$[T|_W]_{\beta_W} = \begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix} \rightsquigarrow f_W(t) = \det\begin{pmatrix} -t & 0 \\ 6 & -t \end{pmatrix} = t^2$$

Find char. poly of T: $\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \underbrace{\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \rightsquigarrow f(t) = \det(A - tI) = t^4$$

Note: $f_W(t) = t^2 \mid t^4 = f(t)$.

Proof of Lemma: Take $\beta_W \subseteq W$ any basis

extend $\rightsquigarrow$ $\beta \subseteq V$ basis

$$\beta = \{\underbrace{\vec{v}_1, \dots, \vec{v}_k}_{\beta \in W}, \vec{v}_{k+1}, \dots, \vec{v}_n\}$$

$$[T]_\beta = \begin{array}{cc} & \begin{array}{cc} k & \end{array} \\ \begin{array}{c} k \\ \\ \end{array} & \begin{pmatrix} \boxed{[T|_W]_{\beta_W}} & * \\ O & \boxed{*} \end{pmatrix} \end{array} \rightsquigarrow \text{compute det.}$$

$\uparrow \because W$ is T-inv.

$$\det\begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det(A)\det(C)$$

$$f(t) = \det\begin{pmatrix} \boxed{} - tI & * \\ 0 & \boxed{*} - tI \end{pmatrix} = \underbrace{\det(\boxed{} - tI)}_{f_W(t)} \cdot \det(\boxed{*} - tI)$$

$\square$

(Caution: $\det\begin{pmatrix} A & B \\ D & C \end{pmatrix} \neq \det(A)\det(C) - \det(B)\det(D)$)

Cayley-Hamilton Theorem:

Let $T: V \to V$ be a linear map. If $f(t) =$ char. poly. of T,

$$\Rightarrow \quad \text{"}f(T) = 0_V\text{"}$$ zero transformation.

Note: $f(t) = t^2 + 2t \rightsquigarrow f(T) = T^2 + 2T : V \to V$ linear.

A matrix example:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \rightsquigarrow f(t) = \det\begin{pmatrix} 1-t & 2 \\ -2 & 1-t \end{pmatrix} = t^2 - 2t + 5$$

"Plug" A into $f(t)$:

$$f(A) = A^2 - 2A + 5I \qquad (= 0 \text{ by Cayley-Hamilton})$$

check by direct calculation

One application: (compute A^{-1})

$$A^2 - 2A + 5I = 0 \underset{\times A^{-1}}{\Rightarrow} A - 2I + 5A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\underbrace{\left(-A+2I\right)}_{\text{easier to compute}}$$

$$[A \mid I] \xrightarrow{\text{rref}} [I \mid A^{-1}]$$